The turbulence tutorial for QFT physicists

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1. Introduction. NS equation and loop calculus

In response to popular demand, I will describe and explain, in the language of QFT, the Navier-Stokes equation and loop calculus step-by-step, leaving nothing implied. I will also provide justifications for the limits involved using the loop's polygonal approximation, where the velocity circulation approximation errors are under control.

1.1. *NS equation in v–ω form*

This is the form of the Navier-Stokes equation we use:

$$\partial_t \vec{v} = -\nu \nabla \times \vec{\omega} - \vec{v} \times \vec{\omega} - \vec{\nabla}\left(p + \frac{\vec{v}^2}{2}\right); \tag{1}$$

$$\nabla \cdot \vec{v} = 0; \tag{2}$$

Here, ν stands for viscosity, $p(\vec{r}, t)$ for pressure, and $\vec{\omega} = \vec{\nabla} \times \vec{v}$ for vorticity. The constant water density is set to one.

1.2. *pressure breaking the locality*

Note that the diffusion term $-\nu \nabla \times \vec{\omega}$ is the same as the equation of motion for QED_3; however, the advection term $-\vec{v} \times \vec{\omega}$ breaks the QED analogy and also breaks locality.

The nonlocal interaction originates from the pressure p, which does not have its own dynamics here, but rather acts as a Lagrange multiplier to enforce incompressibility (the conservation law $\vec{\nabla} \cdot \vec{v}(\vec{r}, t) = 0$ for all t).

By applying $\vec{\nabla}\cdot$ to the NS equation and collecting terms, we find:

$$\vec{\nabla}^2 p = -\vec{\nabla} \cdot (\vec{v} \cdot \vec{\nabla} \vec{v}); \tag{3}$$

$$p = \frac{-1}{\vec{\nabla}^2} \vec{\nabla} \cdot (\vec{v} \cdot \vec{\nabla} \vec{v}); \tag{4}$$

This relation is instantaneous in time (neglecting compressibility, which would otherwise lead to sound waves — e.g., pressure propagation in seawater with a speed of approximately 1500 meters per second).

The pressure can be completely eliminated from the NS equation as follows:

$$\partial_t \vec{v} = -\nu \nabla \times \vec{\omega} - (\vec{v} \times \vec{\omega})_\perp ; \tag{5}$$

$$\vec{V}_\perp \equiv \vec{V} - \frac{1}{\vec{\nabla}^2} \vec{\nabla} \cdot \vec{V} \tag{6}$$

The Laplacian can have zero modes in a finite domain or in spaces with nontrivial topology, but not in the case of infinite space with periodic boundary conditions or simply a constant velocity at infinity.

Thus, the inversion of the Laplacian is unique. It can be written as a Biot–Savart law for the velocity field:

$$\vec{v}(\vec{r}) = \frac{-1}{\vec{\nabla}^2} \vec{\nabla} \times \vec{\omega}(\vec{r}) = \int d^3 r' \frac{\vec{\omega}(\vec{r}') \times (\vec{r}' - \vec{r})}{4\pi |\vec{r} - \vec{r}'|^3} \tag{7}$$

1.3. *The energy and its dissipation*

The energy of the fluid is given by:

$$E = \int_V d^3 r \frac{\vec{v}^2}{2} \tag{8}$$

This energy is not conserved; it is dissipated due to viscosity, arising from friction forces in the fluid. The energy dissipation in an infinite system with periodic or zero-velocity boundary conditions is:

$$-\partial_t E = -\int_V d^3 r \vec{v} \cdot \partial_t \vec{v} = \nu \int_V d^3 r \vec{\omega}^2 \tag{9}$$

This last formula follows from the Navier-Stokes equations and integration by parts using Gauss' theorem. The dropped terms are the divergence of a current:

$$\vec{J}(\vec{r}) = \vec{v}(\vec{r}) \left(p(\vec{r}) + \frac{\vec{v}(\vec{r})^2}{2} \right) + \nu \vec{\omega}(\vec{r}) \times \vec{v}(\vec{r}) \tag{10}$$

The volume integral of the divergence of this current reduces to its flux through the surface of the infinite box V:

$$\int_V d^3 r \vec{\nabla} \cdot \vec{J}(\vec{r}) = \int_{\vec{r} \in \partial V} d\vec{S} \cdot \vec{J}(\vec{r}) \tag{11}$$

This is the energy flux through the boundary, which vanishes under our boundary conditions. If it were present, the energy dissipation would match the net energy flux, so that energy would be conserved (steady turbulence).

We are, however, considering decaying turbulence where the energy flux is absent, and energy is eventually fully dissipated.

2. The loop and the circulation

The **loop** is a periodic function $\vec{C}(\theta) = \vec{C}(\theta + 2\pi)$ mapping the unit circle to $\mathbb{R}^3$ (we use 3D for simplicity, though loop calculus exists in any dimension and also for nonabelian fields).

2.1. *The velocity circulation*

The main dynamical object we study is the circulation and the corresponding abelian loop functional:

$$\Gamma_C[v] = \oint_C \vec{v}(\vec{r},t)\cdot d\vec{r} = \oint \vec{v}(\vec{C}(\theta),t)\cdot \vec{C}'(\theta)d\theta; \tag{12}$$

$$\Psi[C] = \left\langle \exp\left(\imath \frac{\Gamma_C[v]}{\nu}\right)\right\rangle_{NS} \tag{13}$$

The equation for circulation is:

$$\partial_t \Gamma_C[v] = \oint d\vec{C}(\theta)\cdot \partial_t \vec{v} = \oint d\vec{C}(\theta)\left(-\nu\vec{\nabla}\times\vec{\omega} - \vec{v}\times\vec{\omega}\right) \tag{14}$$

The term with the gradient of the so-called enthalpy $H = p + \frac{\vec{v}^2}{2}$ drops from the equation for circulation. The remaining two terms describe diffusion and advection, respectively.

2.2. *Initial data. Thermal fluctuations around a smooth initial velocity field*

The averaging in the loop functional goes over the initial distribution of the velocity field at $t = 0$, which could be, for example, thermal fluctuations around some smooth flow $\vec{v}_0(\vec{r})$:

$$W_0[v] = \exp\left(-\frac{\int d^3r\,(\vec{v}(\vec{r}) - \vec{v}_0(\vec{r}))^2}{2T}\right); \tag{15}$$

$$\Psi_0[C] = \exp\left(\frac{\imath\Gamma_C[v_0]}{\nu} - \frac{T\oint d\vec{C}(\theta)\cdot d\vec{C}(\theta')\delta_\epsilon\left(\vec{C}(\theta) - \vec{C}(\theta')\right)}{2\nu^2}\right); \tag{16}$$

Here $\delta_\epsilon(\vec{r})$ is some regularization of the delta function at molecular distances ϵ.

The last double integral, in the limit $\epsilon \to 0$, reduces to the length of the loop times a factor $\sim 1/\epsilon^2$. This is analogous to a bare mass of a relativistic scalar particle in three dimensions. This can be elaborated further, but we will only need the large-time asymptotic of the solution to the loop equation, which is a fixed point, independent of the initial data.

2.3. *Turbulence as a WKB limit of the loop dynamics*

The viscosity appears in the denominator of the exponential, like Planck's constant in Feynman's path integral representation of quantum mechanics. This analogy is not accidental, as we will see later: the NS equation is equivalent to the Schrödinger equation with $\Psi[C]$ as a wave function in loop space.

For a quantum physicist, it should now be evident that turbulence is a quasiclassical phenomenon. Though nonperturbative, it may still be solvable in appropriate variables, where the circulation plays the role of classical action.

Turbulence appears in flows with circulation much larger than viscosity (the ratio $|\Gamma_C[v]|/\nu$ is the so-called Reynolds number). A large Reynolds number corresponds to strong turbulence; this occurs in a variety of natural phenomena, including plasmas and gases. Even the quark-gluon plasma in the early Universe must have experienced turbulence, although gluodynamics is also involved in that case.

The UV divergences complicate the loop dynamics, so some regularization is needed. This regularization must preserve space symmetries: translations and rotations in $\mathbb{R}^3$.

The simplest is the polygonal approximation of the loop C, with $N \to \infty$ vertices. An alternative would be truncating the Fourier expansion of the periodic function $\vec{C}(\theta)$.

We choose the polygonal approximation because it allows an exact solution at fixed N; the subsequent local limit $N \to \infty$ can also be performed analytically using methods from number theory.

2.4. *Polygonal approximation as regularization*

The polygon r_N is a piecewise linear approximation of that vector function: it is a collection of N points $\vec{r}_k = \vec{C}(2\pi k/N)$ connected to neighbors by straight lines, with the periodicity requirement $\vec{r}_0 = \vec{r}_N$.

The velocity circulation around the polygon, by definition, is:

$$\Gamma_N = \sum_k \int_{\vec{r}_k}^{\vec{r}_{k+1}} d\vec{r} \cdot \vec{v}(\vec{r}); \tag{17}$$

Assuming the velocity field is differentiable (which is necessary to use the NS equation), we can estimate the approximation error as quadratic at a given continuous curve $\vec{C}(\theta)$:

$$\Gamma_N = \Gamma_C[v] + O(1/N^2) \tag{18}$$

Our next task will be to use an identity for the polygonal loop functional:

$$\partial_t \exp\left(\imath \frac{\Gamma_N}{\nu}\right) = O(1/N) + \sum_k \Delta\vec{r}_k \cdot \frac{\partial_t \vec{v}(\vec{r}_k) + \partial_t \vec{v}(\vec{r}_{k-1})}{2} \exp\left(\imath \frac{\Gamma_N}{\nu}\right) =$$
$$\sum_k \Delta\vec{r}_k \cdot \left(-\nu \nabla_k \times \vec{\omega}(\vec{r}_k) - \vec{v}(\vec{r}_k) \times \vec{\omega}(\vec{r}_k)\right) \exp\left(\imath \frac{\Gamma_N}{\nu}\right) \tag{19}$$

We have to replace the vector functions $\vec{\omega}(\vec{r}_k), \vec{v}(\vec{r}_k)$ in this equation with operators in loop space applied to the exponential $\exp\left(\imath \frac{\Gamma_N}{\nu}\right)$.

After that, the operators can be taken outside the averaging over solutions of the NS equation, providing us with the loop equation for the loop functional $\left\langle \exp\left(\imath \frac{\Gamma_N}{\nu}\right) \right\rangle_{NS}$.

3. Gradients and Area Derivative

3.1. *Gradients*

Next, let us consider the derivatives of circulation with respect to one of its vertices. The important observation is that, when all other vertices are fixed, this derivative $\partial_{\vec{r}_k} \equiv \vec{\nabla}_k$ only involves two nearest neighbors.

This is equivalent to differentiating the circulation around the triangle $\triangle(k-1, k, k+1)$ made of three consecutive vertices by the middle vertex $\vec{r}_k$.

Using Stokes' theorem for this triangle with the vector area $(\vec{r}_{k+1} - \vec{r}_{k-1}) \times (\vec{r}_k - \vec{r}_{k-1})/2$, and differentiating this area by $\vec{r}_k$, we have the estimate:

$$\vec{\nabla}_k \Gamma_N = \vec{\omega}(\vec{r}_k) \times (\vec{r}_{k+1} - \vec{r}_{k-1})/2 + O(1/N^2) \tag{20}$$

By itself, this gradient vanishes as $O(1/N)$ due to the factor $\vec{r}_{k+1} - \vec{r}_{k-1} = O(1/N)$.

3.2. *The Area Derivative*

To obtain the area derivative, we need one more gradient. There are many ways to achieve this with accuracy $O(1/N)$, but we choose the most local approximation, meaning that it involves the closest neighbors.

In the limit $N \to \infty$, these approximations are, of course, equivalent. But the specific form we choose simplifies the equations, making them analytically solvable at **finite** N.

We prefer to use another set of variables:

$$\vec{s}_k = \Delta \vec{r}_k; \tag{21}$$

$$\vec{\eta}_k = \partial_{\vec{s}_k}; \tag{22}$$

$$\vec{\nabla}_k = -\Delta \vec{\eta}_{k-1}; \tag{23}$$

The last relation follows from the chain rule:

$$\vec{\nabla}_k = \frac{\partial \vec{s}_k}{\partial \vec{r}_k} \cdot \vec{\eta}_k + \frac{\partial \vec{s}_{k-1}}{\partial \vec{r}_k} \cdot \vec{\eta}_{k-1} = \vec{\eta}_{k-1} - \vec{\eta}_k = -\Delta \vec{\eta}_{k-1} \tag{24}$$

The vorticity can be represented as:

$$\vec{\omega}(\vec{r}_k) = \vec{\eta}_{k-} \times \vec{\nabla}_k \Gamma_N + O(1/N); \tag{25}$$

$$\vec{\eta}_{k-} \equiv \frac{\vec{\eta}_k + \vec{\eta}_{k-1}}{2}; \tag{26}$$

The contour C becomes an open line when we move all $\vec{s}_k$ independently, without enforcing the constraint $\sum \vec{s}_k = 0$. However, the contribution to the time derivative of circulation from the extra gap between the endpoints is:

$$\Delta \partial_t \Gamma \propto H(\vec{r}_N) - H(\vec{r}_0)$$

where $H(\vec{r}) = p(\vec{r}) + \vec{v}^2(\vec{r})/2$ is the enthalpy, which is assumed to be differentiable. Thus, this error term vanishes as we reinstate the closure condition $\sum \vec{s}_k = \vec{r}_N - \vec{r}_0 = 0$.

This discrete formula (with error estimates) is a regularization of the continuum formula:

$$\vec{\omega}(\vec{C}(\theta)) = \frac{1}{2}\left(\frac{\delta}{\delta\vec{C}'(\theta-0)} + \frac{\delta}{\delta\vec{C}'(\theta+0)}\right) \times \int_{\theta-0}^{\theta+0} d\theta' \frac{\delta}{\delta\vec{C}(\theta')}\Gamma_C \tag{27}$$

3.3. *Diffusion Term in the NS Equation*

Now we are in a position to translate the diffusion term $\nu\vec{\nabla}\times\vec{\omega}$ in the NS equation into loop space. Step by step, we have:

$$\nu\left(\vec{\nabla}_k \times \vec{\omega}(\vec{r}_k)\right)\exp\left(\imath\frac{\Gamma_N}{\nu}\right) \rightarrow -\imath\vec{\nabla}_k \times \vec{\eta}_{k-} \times \vec{\nabla}_k \exp\left(\imath\frac{\Gamma_N}{\nu}\right) \tag{28}$$

This relation is based on the observation that the gradient of Γ_N vanishes at $N \to \infty$. Therefore, all gradients apply only to the vorticity factor, which depends on $\vec{r}_k$ directly, not through the circulation.

The gradients applied to $\vec{\omega}(\vec{r}_k)\exp\left(\imath\frac{\Gamma_N}{\nu}\right)$ thus only act on $\vec{\omega}$ in the local limit. This allows us to extend the action of gradients onto the exponential, after which the diffusion term becomes an operator and can be taken out of the averaging in the loop equation:

$$\left\langle \sum_k \Delta\vec{r}_k \cdot (-\nu\nabla_k \times \vec{\omega}(\vec{r}_k))\exp\left(\imath\frac{\Gamma_N}{\nu}\right)\right\rangle \rightarrow \\ \sum_k \Delta\vec{r}_k \cdot \left(-\imath\vec{\nabla}_k \times \vec{\eta}_{k-} \times \vec{\nabla}_k\right)\Psi[C] \tag{29}$$

4. The velocity field and momentum loop equation

4.1. *Velocity field*

Finally, the velocity field at the vertex $\vec{v}(\vec{r}_k)$ can be related to the vorticity through the Biot–Savart law (5):

$$\vec{v}(\vec{r}_k)\exp\left(\frac{\imath\Gamma_N}{\nu}\right) = -\frac{1}{\vec{\nabla}_k^2}\vec{\nabla}_k \times \vec{\omega}(\vec{r}_k)\exp\left(\frac{\imath\Gamma_N}{\nu}\right); \tag{30}$$

Let us verify this relation using the Biot–Savart integral formula for the inverse Laplace operator:

$$\vec{v}(\vec{r}_k)\exp\left(\frac{\imath\Gamma_N}{\nu}\right) = \frac{1}{4\pi}\int d^3r \frac{\vec{r}\times\vec{\omega}(\vec{r}_k+\vec{r})}{|\vec{r}|^3}\exp\left(\frac{\imath\Gamma_N(\vec{r})}{\nu}\right) + \mathcal{O}(1/N); \tag{31}$$

$$\Gamma_N(\vec{r}) = \Gamma_N|_{\vec{r}_k \Rightarrow \vec{r}_k+\vec{r}} \tag{32}$$

At first glance, the loop in the new circulation $\Gamma_N(\vec{r})$ involves two long "wires": $(\vec{r}_{k-1}, \vec{r}_k+\vec{r})$ and $(\vec{r}_k+\vec{r}, \vec{r}_{k+1})$.

However, in the local limit, when the distance $|\vec{r}_{k+1}-\vec{r}_{k-1}| = \mathcal{O}(1/N)$, these two wires have zero area inside the arising thin triangle, so they effectively cancel by the Stokes theorem, assuming the Biot–Savart integral converges:

$$\Gamma_N(\vec{r}) \to \Gamma_N(0) = \Gamma_N \tag{33}$$

This produces the desired result in the exponential of the Biot-Savart formula. There is also an issue of the shifted vorticity $\omega(\vec{r}_k + \vec{r})$ in the BS integral. The operators $\vec{\eta}_{k-} \times \vec{\nabla}_k$ acting on $\exp\left(\imath\Gamma(\vec{r}_k \Rightarrow \vec{r}_k + \vec{r})\right)$ produce for finite r a more complex expression than just $\vec{\omega}(\vec{r}_k + \vec{r})$. In Appendix A, we investigate this expression and prove its exact equivalence to the original Biot-Savart integral, up to a constant (loop-independent) factor. This factor $\Lambda \propto \log V$ diverges when the total volume V of the system goes to infinity, but it does not affect the decaying turbulence solution (Euler ensemble) as the whole $\hat{v} \times \hat{\omega}$ term in the momentum loop equation vanished on this solution.

The convergence of the Biot-Savart integral follows from our boundary conditions, assuming no vorticity at infinity or even the stronger requirement of finite vorticity support.

The phase factor $\exp\left(\imath\Gamma_N(\vec{r})/\nu\right)$ does not influence the absolute convergence, so it can be set to $\exp\left(\frac{\imath\Gamma}{\nu}\right)$ for that purpose and taken out of the integral, returning us to the convergence of the ordinary Biot-Savart integral.

4.2. ***The momentum loop equation***

Therefore, with $\mathcal{O}(1/N)$ accuracy, we can replace the right-hand side of the loop equation by its discrete version with operators involving $\vec{\nabla}_k$:

$$\partial_t \left\langle \exp\left(\frac{\imath\gamma\Gamma}{\nu}\right)\right\rangle = \frac{\imath\gamma}{\nu}\sum_k \Delta\vec{r}_k \cdot \hat{L}_k \left\langle \exp\left(\frac{\imath\gamma\Gamma}{\nu}\right)\right\rangle + \mathcal{O}(1/N); \tag{34a}$$

$$\hat{L}_k = -\nu\vec{\nabla}_k \times \hat{\omega}_k + \hat{\omega}_k \times \hat{v}_k; \tag{34b}$$

$$\hat{v}_k = -\frac{1}{\vec{\nabla}_k^2}\vec{\nabla}_k \times \hat{\omega}_k; \tag{34c}$$

$$\hat{\omega}_k = \frac{\imath\gamma}{\nu}\vec{\eta}_{k-} \times \vec{\nabla}_k; \tag{34d}$$

The loop operator $\hat{L}$ in (34) dramatically simplifies in the functional Fourier space, which we call momentum loop space. In our discrete approximation, the momentum loop will also be a polygon with N sides.

The origin of this simplification is the lack of direct dependence of the loop operator $\hat{L}(\theta)$ on the loop C itself. Only derivatives $\vec{\nabla}_k, \vec{\eta}_k$ enter this operator.

From the point of view of quantum mechanics in loop space, our Hamiltonian depends only on canonical momenta but not on canonical coordinates. This property is exact as long as we do not add external forces.

This remarkable symmetry (translational invariance in loop space) allows us to look for the "superposition of plane waves" Ansatz:

$$\Psi(C|t) = \langle \psi_p(t) \rangle_{\text{init}}\,; \tag{35a}$$

$$\psi_p(t) = \exp\left(\frac{\imath}{\nu}\sum_k \Delta\vec{r}_k \cdot \vec{P}_k(t)\right) \tag{35b}$$

Here, the averaging $\langle \dots \rangle_{\text{init}}$ goes over all trajectories $\vec{P}_k(t)$ starting from random initial data $\vec{P}_k(0)$, distributed with the appropriate probability to reproduce the initial value $\Psi(C|0)$. We will discuss this initial distribution in the next sections.

The operators $\vec{\nabla}_k, \vec{\eta}_k$ become ordinary vectors when applied to ψ_p in (35):

$$\vec{\nabla}_k \psi_p = -\frac{\imath}{\nu}\Delta\vec{P}_{k-1}\psi_p; \tag{36a}$$

$$\vec{\eta}_{k-}\psi_p = \frac{\imath}{\nu}\vec{P}_{k-}\psi_p; \tag{36b}$$

$$\vec{P}_{k-} \equiv \frac{\vec{P}_k + \vec{P}_{k-1}}{2}; \tag{36c}$$

$$\hat{\omega}_k \propto \frac{\imath}{\nu}\vec{P}_{k-} \times \Delta\vec{P}_k \tag{36d}$$

The velocity circulation can be rewritten up to $\mathcal{O}(1/N)$ corrections as a symmetric sum:

$$\sum_k \Delta\vec{r}_k \cdot \vec{P}_k(t) + \mathcal{O}(1/N) = \sum_k \frac{\Delta\vec{r}_k + \Delta\vec{r}_{k+1}}{2} \cdot \vec{P}_k(t) = \sum_k \Delta\vec{r}_k \cdot \vec{P}_{k-}(t) \tag{37}$$

We did not assume here anything about the continuity of $\vec{P}_k$; we only assumed that $|\Delta\vec{r}_{k+1} - \Delta\vec{r}_k| \ll |\Delta\vec{r}_k|$, which is true for a smooth loop.

5. The momentum loop equation (MLE) and its continuum limit

The discrete loop equation (34) with our Ansatz (35), after some algebraic transformations using the identities (36) and (37), reduces to the following momentum loop equation (MLE):

$$\nu\partial_t\vec{P} = -(\Delta\vec{P})^2\vec{P} + \Delta\vec{P}\left(\vec{P}\cdot\Delta\vec{P} + \imath\left(\frac{(\vec{P}\cdot\Delta\vec{P})^2}{\Delta\vec{P}^2} - \vec{P}^2\right)\right); \tag{38a}$$

$$\Delta\vec{P} \equiv \vec{P}_k - \vec{P}_{k-1}; \tag{38b}$$

$$\vec{P} \equiv \vec{P}_{k-} \tag{38c}$$

In the local limit $N \to \infty$, the momentum loop develops a discontinuity $\Delta\vec{P}(\theta)$ at every point $0 < \theta \le 2\pi$, making it a fractal curve in complex space $\mathbb{C}_d$. Such a curve can only be defined using a limiting process, such as a polygonal approximation or a Fourier expansion of a periodic function of θ with slowly decaying Fourier coefficients.

You can regard this curve as a periodic random process hopping around the circle — more about this process will follow in the context of decaying turbulence.

The details can be found in Refs.[1–3] We will omit the arguments t, k in these loop equations going forward, as the equations have no explicit dependence on either parameter.

6. Universality and Scaling of MLE

Various symmetry properties affect the solution space of the momentum loop equation (MLE), especially its fixed trajectories.

First of all, this equation is parametrically invariant:

$$\vec{P}(\theta, t) \Rightarrow \vec{P}(f(\theta), t), \quad f'(\theta) > 0; \tag{39}$$

Naturally, any initial condition $\vec{P}(\theta, 0) = \vec{P}_0(\theta)$ will break this invariance. Each such initial condition will generate a family of solutions corresponding to initial data $\vec{P}_0(f(\theta))$.

The absence of explicit time dependence on the right-hand side leads to time translation invariance:

$$\vec{P}(\theta, t) \Rightarrow \vec{P}(\theta, t + a) \tag{40}$$

Less trivial but also significant is the time-rescaling symmetry:

$$\vec{P}(\theta, t) \Rightarrow \sqrt{\lambda}\, \vec{P}(\theta, \lambda t) \tag{41}$$

This symmetry follows because the right-hand side of (38) is a homogeneous functional of degree three in $\vec{P}$, with no explicit time dependence.

This scaling transformation is quite different from that of the original Navier-Stokes equation, which involves rescaling the viscosity parameter:

$$\vec{v}(\vec{r}, t) \Rightarrow \frac{\vec{v}(\alpha \vec{r}, \lambda t)}{\alpha \lambda}; \tag{42}$$

$$\nu \Rightarrow \nu \frac{\alpha^2}{\lambda} \tag{43}$$

In our case, there is a genuine scale invariance without any parameter change — in other words, there are no remaining dimensional parameters in MLE.

Using this invariance, one can define the following transformation of the momentum loop and its variables:

$$\vec{P} = \sqrt{\frac{\nu}{2(t + t_0)}}\vec{F} \tag{44}$$

The new vector function $\vec{F}$ satisfies the following dimensionless equation:

$$2\partial_\tau \vec{F} = \left(1 - (\Delta \vec{F})^2\right)\vec{F} + \Delta \vec{F}\left(\vec{F} \cdot \Delta \vec{F} + \imath\left(\frac{(\vec{F} \cdot \Delta \vec{F})^2}{\Delta \vec{F}^2} - \vec{F}^2\right)\right); \tag{45}$$

$$\tau = \log\left(\frac{t + t_0}{t_0}\right) \tag{46}$$

The **viscosity has disappeared from this equation**; it now enters only the initial data:

$$\vec{F}(\theta, 0) = \sqrt{\frac{2t_0}{\nu}} \vec{P}_0(\theta) \tag{47}$$

This universality property is extremely important.

Note that the loop functional is now represented as:

$$\Psi(C, t) = \left\langle \exp\left(\frac{\imath \oint d\vec{C}(\theta) \cdot \vec{F}\left(\theta, \log\frac{t+t_0}{t_0}\right)}{\sqrt{2\nu(t+t_0)}} \right) \right\rangle \tag{48}$$

with the square root of viscosity in the denominator as a coupling constant in a nonlinear QFT. The averaging $\langle \dots \rangle$ goes over the manifold of solutions $\vec{F}(\theta, \tau)$ of the ODE (45).

This formula immediately suggests that turbulence is a quasiclassical phenomenon in our quantum mechanical system, which can be studied using the well-known WKB method.

In the conventional approach to fluid mechanics, based on the Navier-Stokes equation, the Reynolds number — which distinguishes between laminar and turbulent flow — appears in the equation and controls the relative magnitude of nonlinearity. One must study various regimes in that equation, including the inviscid limit (presumably related to turbulence but different from the Euler equation due to the dissipation anomaly).

In our dual theory, representing the same Navier-Stokes dynamics as a quantum system, the dynamical equation (45) is universal; it does not depend on the Reynolds number.

This number enters through the initial data and the relation between our solution $\vec{F}$ and the loop functional (i.e., the PDF for the circulation as a functional of the loop shape).

The evolution of the loop functional Ψ inside the unit circle in the complex plane proceeds along universal trajectories determined by (45). The Reynolds number describes the initial position of Ψ inside the circle. The distance $|\Psi - 1|$ from the fixed point $\Psi_* = 1$ is the true measure of turbulence. One could expect a laminar flow solution in a small neighborhood of this fixed point (corresponding to potential flow).

7. Decaying turbulence

The solutions originating deep inside the unit circle, far from $\Psi = 1$, can become turbulent and eventually decay to $\Psi \to 1$ due to energy dissipation by vorticity structures. This decay for $\vec{P}(\theta, t)$ corresponds to the fixed point equation for $\vec{F}$:

$$\left((\Delta\vec{F})^2 - 1\right)\vec{F} = \Delta\vec{F}\left(\gamma^2 \vec{F}\cdot\Delta\vec{F} + \imath\gamma\left(\frac{(\vec{F}\cdot\Delta\vec{F})^2}{\Delta\vec{F}^2} - \vec{F}^2\right)\right) \tag{49}$$

This fixed point $\vec{F}(\theta)$ is not itself a solution of the Cauchy problem for the loop functional, although we expect some solutions to asymptotically approach it as $t \to \infty$.

This fixed point represents the solution of the loop equation with the boundary condition $\Psi(\theta, +\infty) = 1$, which describes the flow eventually stopping due to dissipation of kinetic energy:

$$E = \int d^3 r \frac{\vec{v}^2}{2}, \quad \partial_t E = -\nu \int d^3 r \, \vec{\omega}^2 < 0.$$

7.1. *Fixed point solution*

The saddle point equation (49) was solved and investigated in previous works.[1,2] The solution $\vec{F}(\theta)$ is a fractal curve defined as the $N \to \infty$ limit of a polygon $\vec{F}_0, \dots, \vec{F}_N = \vec{F}_0$ with vertices:

$$\vec{F}_k = \Omega \cdot \frac{\left\{\cos(\alpha_k), \sin(\alpha_k), \imath \cos\left(\frac{\beta}{2}\right)\right\}}{2 \sin\left(\frac{\beta}{2}\right)}; \tag{50}$$

$$\theta_k = \frac{k}{N}, \quad \beta = \frac{2\pi p}{q}, \quad N \to \infty; \tag{51}$$

$$\alpha_k = \alpha_{k-1} + \sigma_k \beta, \quad \sigma_k = \pm 1, \quad \beta \sum \sigma_k = 2\pi p r; \tag{52}$$

$$\Omega \in SO(3) \tag{53}$$

The parameters $\Omega, p, q, r, \sigma_0, \dots, \sigma_N = \sigma_0$ are random, making the solution $\vec{F}(\theta)$ a fixed *manifold* rather than a point. We referred to this manifold as the "big Euler ensemble" in.[1]

This is a fixed point of (45) under the discrete form of the derivative:

$$\Delta \vec{F} = \vec{F}_k - \vec{F}_{k-1}; \tag{54}$$

$$\vec{F} = \frac{\vec{F}_k + \vec{F}_{k-1}}{2} \tag{55}$$

The right-hand side of (45) vanishes — both the coefficient in front of $\Delta\vec{F}$ and the one in front of $\vec{F}$ are zero. Otherwise, we would have $\vec{F} \parallel \Delta\vec{F}$, leading to vanishing vorticity.[1]

This condition leads to two scalar equations:

$$(\Delta \vec{F})^2 = 1; \tag{56a}$$

$$\vec{F}^2 - \frac{1}{4} = \left(\vec{F} \cdot \Delta\vec{F} - \frac{\imath}{2}\right)^2 \tag{56b}$$

The discrete Ising variables $\sigma_k = \pm 1$ emerge from the structure of the solution. The requirement that $\beta = \frac{2\pi p}{q}$ be rational comes from the periodicity condition — we'll prove this explicitly later.

In the polygonal limit, the circulation integral becomes a Lebesgue-type sum:

$$\oint d\vec{C}(\theta) \cdot \vec{F}(\theta) \to \sum_k \Delta \vec{C}_k \cdot \vec{F}_k \tag{57}$$

A remarkable property of this solution is that even though $\vec{F}_k$ has an imaginary part, the resulting circulation (57) is real! The imaginary part of $\vec{F}_k$ is constant and therefore cancels due to loop closure:

$$\sum_k \Delta \vec{C}_k = 0.$$

Furthermore, $-\vec{F}_k$ and $\vec{F}_k$ are symmetrically distributed due to the integration over the rotation matrix Ω. This symmetry — a π rotation in the xy-plane combined with complex conjugation — leaves the loop functional invariant.

As a result, the probability distribution function (PDF) of the circulation $\Gamma = \sum_k \Delta \vec{C}_k \cdot \vec{P}_k$ is an even function. However, multiple-vorticity correlators are not suppressed by this symmetry. The area derivative $\hat{\omega}_k$ is quadratic in $\vec{P}$ and hence not sign-reversing.

Thus, odd correlators such as $\omega^\alpha(1)\omega^\beta(2)\omega^\gamma(3)$ may still be nonzero. Corresponding triple velocity correlators (e.g., $\vec{v}^\alpha(1)\vec{v}^\beta(2)\vec{v}^\gamma(3)$) follow from Fourier analysis, where $\vec{v}_k = \imath\vec{k} \times \vec{\omega}_k/\vec{k}^2$, up to purely potential terms linear in $\vec{r}$.

These potential terms do not contribute to energy flux in wavevector space. Contrary to popular belief — as discussed in[2] — they result in gradients of delta functions, $\partial_{\vec{k}}\delta^3(\vec{k})$, not a constant flux.

Such terms depend on boundary conditions and do not represent spontaneous stochasticity caused by internal vorticity structures.

7.2. *The proof of the Euler ensemble as a fixed point of MLE*

Let us present here the proof of this solution, verified by *Mathematica®* (see[4]).

Theorem 1. *The Euler ensemble solves the discrete MLE.*

Proof. We start from the general Ansatz with real vectors $\vec{A}, \vec{f}_k$, corresponding to the real circulation in (57):

$$\vec{F}_k = \imath\vec{A} + \frac{\vec{f}_{k-1} + \vec{f}_k}{2}; \tag{58}$$

$$\Delta\vec{F}_k = \vec{f}_k - \vec{f}_{k-1}; \tag{59}$$

$$(\vec{f}_k - \vec{f}_{k-1})^2 = 1 \tag{60}$$

Analyzing the imaginary and real parts of the second equation in (56), we observe that the imaginary part vanishes provided:

$$\vec{A} \cdot \vec{f}_k = 0 \quad \forall k; \tag{61}$$

$$\vec{f}_k^2 = \vec{f}_{k-1}^2 \quad \forall k \tag{62}$$

We conclude that $\vec{f}_k$ lies on a circle of radius R centered at the origin of the plane orthogonal to $\vec{A}$.

In the coordinate frame where $\vec{A} = \{0, 0, A\}$, we can write:

$$\vec{f}_k = R\{\cos(\alpha_k), \sin(\alpha_k), 0\} \tag{63}$$

The $SO(3)$ matrix that rotates these vectors into this coordinate frame can be absorbed into the overall rotation matrix Ω in our solution (A.10).

The radius R and the magnitude $A = |\vec{A}|$ are determined by the real part of our equations:

$$4A^2 = 2R^2(1 + \cos(\alpha_k - \alpha_{k-1})); \tag{64a}$$

$$1 = 2R^2(1 - \cos(\alpha_k - \alpha_{k-1})) \tag{64b}$$

Solving these equations gives:

$$\alpha_k = \alpha_{k-1} + \beta\sigma_k, \quad \sigma_k^2 = 1; \tag{65}$$

$$R = \frac{1}{2\sin(\beta/2)}; \quad A = \frac{1}{2\tan(\beta/2)} \tag{66}$$

The periodicity of the sequence $\vec{f}_k$ requires the angular step to be a rational fraction of 2π, which brings us directly to the Euler ensemble (A.10). □

7.3. *Euler ensemble as a random walk on a regular star polygon*

Geometrically, the vertices $\vec{f}_k$ lie on a regular star polygon with q sides of unit length, with vertices located at $R\exp(()\imath k\beta)$, $k = 1, \dots, q$. These polygons were classified by Thomas Bradwardine (Archbishop of Canterbury) and later by Johannes Kepler. They are denoted $\{q/p\}$ (the Schläfli symbol).

We show several examples in Fig. 1. A general star polygon is characterized by co-prime integers $p < q < N$, with $N - q \equiv 0 \mod 2$.

Euler totient functions count these polygons. The number $N > q$ counts how many times the sequence $\vec{f}_k$ covers the same polygon — that is, each vertex of the star may be visited multiple times.

The Ising variables σ_k describe a random walk around the polygon, with the constraint that it returns to the starting point after N steps. The walk proceeds step-by-step via $k \leftrightarrow k + 1$, and the sign of σ_k determines the step direction.

The periodicity condition requires that β be a rational multiple of 2π, thus quantizing the angle and the radius. This introduces number theory into the statistical structure.

Each polygon may be covered several times, and the total winding number w is given by:

$$w = \frac{p}{q}\sum_1^N \sigma_k = pr$$

Surprisingly, this fundamental random walk problem on a 500-year-old geometric manifold has only now been solved.

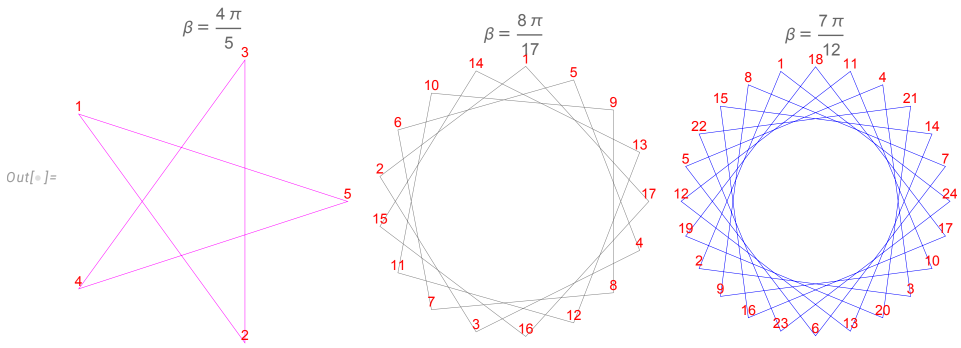

Figure 1. Regular star polygons for Euler ensembles with various p, q. The σ_k variable indicates the direction of the random step along link $k \leftrightarrow k+1$. The walk may wrap around the polygon multiple times, as long as it returns to its start.

7.4. *Euler ensemble as string theory with discrete target space*

This random walk can also be interpreted as a closed fermionic string with discrete target space made of regular star polygons lying in randomly rotated planes.

Integrating over fermionic degrees of freedom in the quantum trace of the evolution operator is equivalent to summing over occupation numbers $n_k = 0, 1$, which correspond to $\sigma_k = 2n_k - 1$, giving the random walk directions.

The target space is the set of regular star polygons $\{q/p\}$, with vertex coordinates $\vec{f}_k$.

Integration over the target space becomes a discrete sum over Euler ensemble states: - Rational angles $\frac{p}{q}$ - Ising configurations $\nu_k = 0, 1$ - Winding number $w = \frac{p}{q} \sum (2\nu_k - 1)$

Placing the polygons (for fixed N) on a torus in 3D, ordered by angle β, gives the worldsheet of the discrete string — shown in Fig. 2. The red/green coloring of the edges indicates the random directions of the walk (or equivalently, fermionic occupation).

The large disk (which becomes infinite as $N \to \infty$) corresponds to the endpoints $\beta = \frac{2\pi}{N}, \frac{2\pi(N-1)}{N}$.

The solution of the Euler ensemble[1] is based on new number-theoretic identities involving cotangent sums of π-fractions. These identities relate the sums to Jordan multi-totient functions weighted with Bernoulli coefficients.

The key nontrivial formula is (48), which connects this ensemble to the observable loop functional of decaying turbulence.

In string theory language, where the momentum loop is the target space, and fermionic occupation is encoded by σ_k, this formula is the dual amplitude of a string theory with discrete target space. The quantity

$$\frac{\Delta \vec{C}(\theta)}{\sqrt{2\nu(t+t_0)}}$$

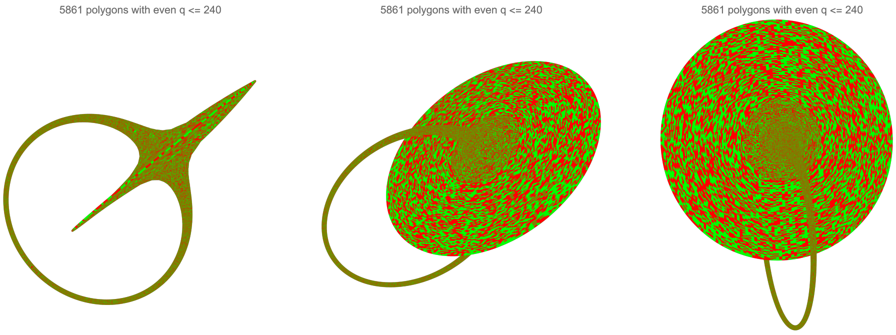

Figure 2. The worldsheet of the discrete string built from regular star polygons with unit side. Red/green edge colors indicate random walk direction (i.e., fermionic occupation).

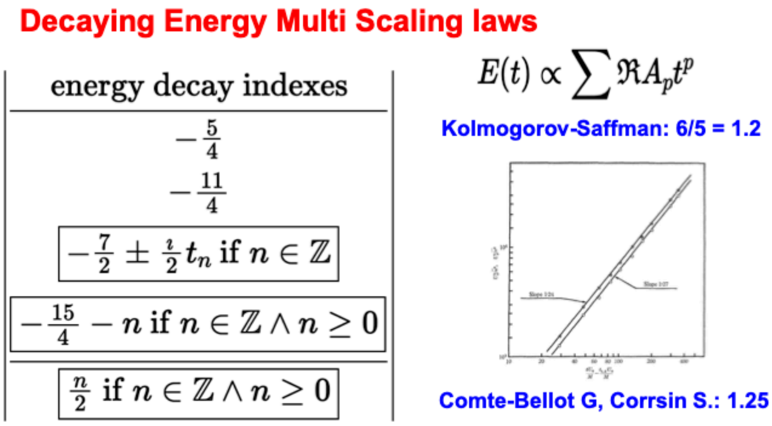

Figure 3. Predictions of this theory compared with classical grid turbulence decay data (1966). The data significantly deviates from the Kolmogorov-Saffman prediction (1.20) but aligns precisely with our predicted index $5/4 = 1.25$.

plays the role of external momentum distributed along the string's position $\vec{F}(\theta)$.

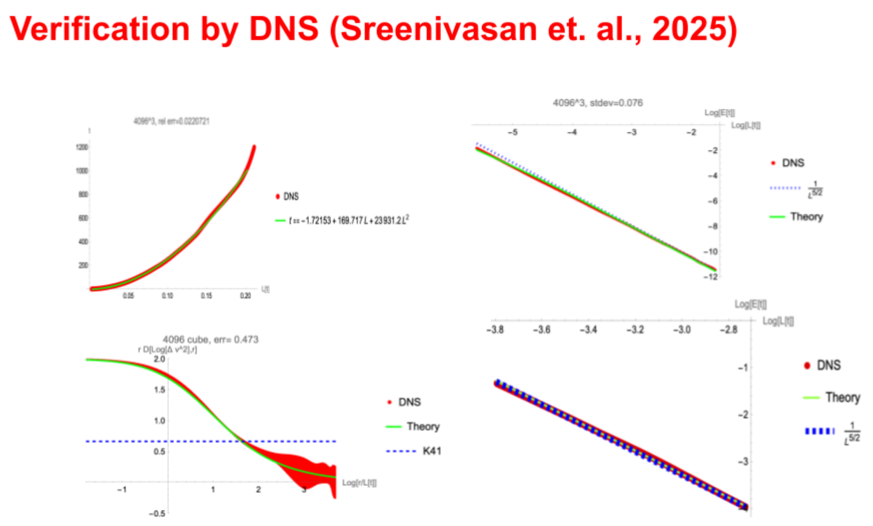

Figure 4. Raw data from 2025 DNS (A. Rodhiya and K.R. Sreenivasan, 4096^3) and 2024 DNS (J.J. Panicacheril, D. Donzis, K.R. Sreenivasan, 1024^3). The effective scaling index of the second velocity moment is plotted vs. $\log r/\sqrt{t+t_0}$ (lower-left). Both datasets match the theoretical curve (green) within error bars. Kolmogorov prediction (2/3) shown as dashed black line — clearly invalid. Upper left: $L(t)$ vs. prediction $L = \sqrt{t+t_0}$. Right panels: decaying energy compared to $E \propto L^{-5/2}$. Our curve (green) fits DNS data perfectly, including subleading corrections.

7.5. *The turbulence/string duality and final insights*

This turbulence/string duality reveals the hidden beauty of primes beneath the chaotic surface of turbulence.

The corresponding universal energy spectrum for decaying turbulence was computed in closed form in,[2] in the quasiclassical limit $\nu = \tilde{\nu}/N^2 \to 0$. It closely matches both experimental and numerical DNS data.

A detailed comparison with real and numerical experiments in decaying turbulence was published in.[2,5]

Let us recall two figures from that work: Fig.3 and Fig.4.

This theory is the first to derive a complete, exact, and universal solution to 3D decaying turbulence — grounded in duality, number theory, and quantum field theory in loop space.

Best regards,

Sasha

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Appendix A. The estimate of the velocity term with line integral over polygon

Here is a revised estimate of the velocity term in the loop equation. We find a non-universal normalization factor, which fortunately drops from the Euler ensemble solution.

We use line integrals over polygons instead of discrete sums

$$\Gamma = \sum_k \int_{r_k}^{r_{k+1}} d\vec{r} \cdot \vec{v}(r) \tag{A.1}$$

The computations are based on the following two lemmas:

Lemma Appendix A.1. *Circulation over the polygon $P : r_1, \dots r_N$ equals circulation of the polygon P' with skipped vertex r_k plus area integral over triangle $\triangle(r_{k-1}, r_k, r_{k+1})$ of the normal vorticity.*

Proof. Follows from the Stokes theorem after adding a backtracking line integral between r_{k-1}, r_{k+1} □

Lemma Appendix A.2. *The area integral $\Gamma_\triangle$ over triangle $\triangle(r_{k-1}, r_k + \rho, r_{k+1})$ of the normal vorticity has the following asymptotic expression in the limit when $a = r_{k+1} - r_{k-1} = O(1/N) \to 0$:*

$$\Gamma_\triangle \to \vec{\rho} \times \vec{a} \cdot \int_0^1 d\lambda (1 - \lambda) \vec{\omega}(r_k + \lambda \vec{\rho}) + O(1/N^2)$$

, where we only assumed $\vec{a} \sim O(1/N)$.

Proof. Let us use the frame where $\vec{\rho} = \{\rho, 0, 0\}$, and $a_z = 0$, so that triangle is in x, y plane. Up to the $O(1/N^2)$ terms, we could replace the small side of the triangle by its y component, after which we have an equilateral triangle $O(1/N^2)$. Let us consider the line from the far corner to the middle of this side $r = \bar{C} + \lambda \rho$. The area integral can be written as

$$\int_0^\rho dx \int_{-\alpha(\rho - x)}^{\alpha(\rho - x)} dy \omega_n(\bar{C} + (x, y)) \tag{A.2}$$

with $\alpha = a_y/\rho$. Expanding vorticity in y we see that the linear terms cancel by symmetry in the y integral, and higher terms do not contribute to the required $O(1/N)$ terms. Integrating by y and changing $x = \lambda\rho$, we find the quoted result, after transforming ti a general frame. □

Corollary Appendix A.1. *The derivative of the circulation by* $\vec{a}$ *can be written as*

$$-\partial_{\vec{a}}\Gamma_{\triangle} \to \vec{\rho} \times \int_0^1 d\lambda(1-\lambda)\vec{\omega}(r_k + \lambda\vec{\rho}) + O(1/N),$$

Proof. Straightforward. □

The BS integral can now be written up to $O(1/N)$ as

$$\begin{aligned} -\imath \int \frac{d^3\rho}{|\rho|^3}\rho \times (\partial_{\vec{a}} \times \partial_{\vec{\rho}}) \exp(\imath\Gamma) \to \\ \exp(\imath\Gamma) \int \frac{d^3\rho}{|\rho|^3}\vec{\rho} \times \left(\partial_{\vec{\rho}} \times \left(\vec{\rho} \times \int_0^1 d\lambda(1-\lambda)\vec{\omega}(r_k + \lambda\vec{\rho})\right)\right) \end{aligned} \tag{A.3}$$

Now, assuming the integration over ρ converges we can rescale $\rho \Rightarrow \rho/\lambda$ and change integration order:

$$\begin{aligned} \int \frac{d^3\rho}{|\rho|^3}\vec{\rho} \times \left(\partial_{\vec{\rho}} \times \left(\vec{\rho} \times \int_0^1 d\lambda(1-\lambda)\vec{\omega}(r_k + \lambda\vec{\rho})\right)\right) = \\ \int_0^1 d\lambda(1-\lambda)/\lambda \int \frac{d^3\rho}{|\rho|^3}\vec{\rho} \times (\partial_{\vec{\rho}} \times (\vec{\rho} \times \vec{\omega}(r_k + \vec{\rho}))) \end{aligned} \tag{A.4}$$

In the resulting ρ integral we can replace $\partial_{\vec{\rho}}$ as $-\overleftarrow{\partial_{\vec{\rho}}}$ acting on left factors $\vec{\rho}/|\vec{\rho}|^3$. Performing the gradients and tensor sums we arrive at the same BS integral, up to a normalization factor

$$\begin{aligned} \int_0^1 d\lambda(1-\lambda)/\lambda \int \frac{d^3\rho}{|\rho|^3}\vec{\rho} \times \left(\overleftarrow{\partial_{\vec{\rho}}} \times (\vec{\rho} \times \vec{\omega}(r_k + \vec{\rho}))\right) = \\ \Lambda \int \frac{d^3\rho}{|\rho|^3}\vec{\rho} \times \vec{\omega}(r_k + \vec{\rho}); \end{aligned} \tag{A.5}$$

$$\Lambda = \int_\epsilon^1 d\lambda(1-\lambda)/\lambda \to \log(1/\epsilon) \tag{A.6}$$

This logarithmic divergence came from the large distances ρ in the initial integral. At small λ, the BS kernel linearly grows with ρ, but ω does not provide convergence if $\lambda \sim 1/\rho \to 0$.

In short, this factor is influenced by the finite size effects in the large volume limit. Fortunately, the coefficient in front of the logarithm is universal; the finite size effects affect only the value of ϵ.

We conclude that the advection term in the NS equation will be correctly reproduced provided the corresponding term in the discrete loop equation is divided by Λ. This leads to the renormalized MLE

$$\nu \partial_t \vec{P} = -\gamma^2 (\Delta \vec{P})^2 \vec{P} + \Delta \vec{P} \left(\gamma^2 \vec{P} \cdot \Delta \vec{P} + \frac{\imath \gamma}{\Lambda} \left(\frac{(\vec{P} \cdot \Delta \vec{P})^2}{\Delta \vec{P}^2} - \vec{P}^2 \right) \right); \quad \text{(A.7)}$$

$$\Delta \vec{P} \equiv \vec{P}_k - \vec{P}_{k-1}; \quad \text{(A.8)}$$

$$\vec{P} \equiv \vec{P}_{k-} \quad \text{(A.9)}$$

However, the renormalization factor Λ drops from the Euler ensemble solution, as does the factor γ. I verified this by direct computation with the extra factor present.

$$\vec{P} = \sqrt{\frac{\nu}{2(t + t_0)}} \frac{\vec{F}}{\gamma}; \quad \text{(A.10)}$$

$$\vec{F}_k = \Omega \cdot \frac{\left\{ \cos(\alpha_k), \sin(\alpha_k), \imath \cos\left(\frac{\beta}{2}\right) \right\}}{2 \sin\left(\frac{\beta}{2}\right)}; \quad \text{(A.11)}$$

$$\theta_k = \frac{k}{N}; \; \beta = \frac{2\pi p}{q}; \; N \to \infty; \quad \text{(A.12)}$$

$$\alpha_k = \alpha_{k-1} + \sigma_k \beta; \; \sigma_k = \pm 1, \; \beta \sum \sigma_k = 2\pi p r; \quad \text{(A.13)}$$

$$\Omega \in SO(3) \quad \text{(A.14)}$$